

# Option Pricing Formulas based on a non-Gaussian Stock Price Model

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February 1, 2008

## Abstract

Options are financial instruments that depend on the underlying stock. We explain their non-Gaussian fluctuations using the nonextensive thermodynamics parameter  $q$ . A generalized form of the Black-Scholes (B-S) partial differential equation, and some closed-form solutions are obtained. The standard B-S equation ( $q = 1$ ) which is used by economists to calculate option prices requires multiple values of the stock volatility (known as the volatility smile). Using  $q = 1.5$  which well models the empirical distribution of returns, we get a good description of option prices using a single volatility.

Although empirical stock price returns clearly do not follow the log-normal distribution, many of the most famous results of mathematical finance are based on that distribution. For example, Black and Scholes (B-S) [1] were able to derive the prices of options and other derivatives of the underlying stock based on such a model. An option is the right to buy or sell the underlying stock at some set price (called the strike) at some time in the future. While of great importance and widely used, such theoretical option prices do not quite match the observed ones. In particular, the B-S model underestimates the prices of options in situations when the stock price at the time of exercise is different from the strike. In order to match the observed

market values, the B-S model would need to use a different value of the volatility for each value of the strike. Such “implied volatilities” of options of various strike prices form a convex function known as the “volatility smile”.

Indeed, attempts have been made to modify the B-S model in ways that can correct for the smile effect (cf [2] or more recently [3, 4]). However, those approaches are often very complicated or rather ad-hoc, and do not result in manageable closed form solutions, which is the forte of the B-S approach. In this paper we do however succeed in developing a theory of non-Gaussian option pricing which allows for closed form solutions for European options, which are such that can be exercised exclusively on a fixed day of expiration and not before (as is the case for American options).

Our approach uses stochastic processes with statistical feedback [5] as a model for stock prices. Such processes were recently developed within the Tsallis generalized thermostatics [6]. The driving noise can be interpreted as a generalized Wiener process governed by a Tsallis distribution of entropic index  $q$ . In the limit  $q \rightarrow 1$  the standard model is recovered. For  $q \approx 1.5$ , this model closely fits the empirically observed distribution for many financial time series, such as stock prices [7] (Figure 1), SP500 index, [7][8], FX rates, etc. This is consistent with a cumulative distribution having power tails of index 3 [9]. We derive closed form option pricing formulas, reproducing prices which, relative to the standard B-S model, exhibit volatility smiles very close to those observed empirically (Figure 4). Note that  $q = 1.5$  well models hydrodynamic turbulence on small scales [10], reinforcing notions of a possible analogy between these two systems.

The standard model for stock prices is  $S = S_0 e^{Y(t)}$  where  $Y(t) = \ln(S(t + t_0)/\ln S(t_0))$  follows

$$dY = \mu dt + \sigma d\omega \quad (1)$$

The drift  $\mu$  is the mean rate of return and  $\sigma^2$  is the variance of the stock logarithmic return. The noise  $\omega$  is a Brownian motion defined with respect to a probability measure  $F$ . It is a Wiener process and satisfies  $E^F[d\omega(t)d\omega(t')] = dt dt' \delta(t - t')$  where the notation  $E^F[\ ]$  means the expectation value with respect to the measure  $F$ . This model yields a Gaussian distribution for  $Y$  resulting in a log-normal distribution for  $S$ . Within this framework, Black and Scholes were able to establish a pricing model to obtain the fair value of options on the underlying stock  $S$ .

In this paper we assume that the log returns  $Y(t) = \ln S(t + t_0)/\ln S(t_0)$

follow

$$dY = \mu dt + \sigma d\Omega \quad (2)$$

with respect to the timescale  $t$ . Here  $\Omega$  evolves according to the statistical feedback process[5]

$$d\Omega = P(\Omega)^{\frac{1-q}{2}} d\omega \quad (3)$$

The probability distribution  $P$  satisfies the nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t} P(\Omega, t \mid \Omega', t') = \frac{1}{2} \frac{\partial}{\partial \Omega^2} P^{2-q}(\Omega, t \mid \Omega', t') \quad (4)$$

Explicit solutions for  $P$  are given by Tsallis distributions [11]

$$P_q(\Omega, t \mid \Omega(0), 0) = \frac{1}{Z(t)} \{1 - \beta(t)(1-q)[\Omega(t) - \Omega(0)]^2\}^{\frac{1}{1-q}} \quad (5)$$

Choosing  $\beta(t) = c^{\frac{1-q}{3-q}} ((2-q)(3-q)t)^{-2/(3-q)}$  and  $Z(t) = ((2-q)(3-q)ct)^{\frac{1}{3-q}}$  ensures that the initial condition  $P_q = \delta(\Omega(t) - \Omega(0))$  is satisfied. The  $q$ -dependent constant  $c$  is given by  $c = \beta Z^2$  with  $Z = \int_{-\infty}^{\infty} (1 - (1-q)\beta\Omega^2)^{\frac{1}{1-q}} d\Omega$  for any  $\beta$ . With  $\Omega(0) = 0$ , we obtain a generalized Wiener process, distributed according to a zero-mean Tsallis distribution. In the limit  $q \rightarrow 1$  the standard theory Eq(1) is recovered, and  $P_q$  becomes a Gaussian. We are concerned with the range  $1 \leq q < 5/3$  in which positive tails and finite variances are found [12]. The distribution for  $\ln S$  becomes

$$P_q(\ln S(t+t_0), t+t_0 \mid \ln S(t_0), t_0) = \frac{1}{Z(t)} \{1 - \tilde{\beta}(t)(1-q)[\ln \frac{S(t+t_0)}{S(t_0)} - \mu t]^2\}^{\frac{1}{1-q}} \quad (6)$$

with  $\tilde{\beta} = \beta(t)/\sigma^2$ . This implies that log-returns  $\ln[S(t+t_0)/S(t_0)]$  over the timescale  $t$  follow a Tsallis distribution, consistent with empirical evidence for several markets, e.g. S&P500 (Figure 1 [7]) [8], with  $q \approx 1.5$ .

Our model exhibits a feedback from the macroscopic level characterised by  $P$ , to the microscopic level characterised by  $\Omega$ . We can imagine that this is really due to the interactions of many individual traders whose actions all contribute to shocks to the stock price which keep it in equilibrium. Their collective behaviour yields a nonhomogenous reaction to returns: rare events (i.e. extreme returns) will be accompanied by large reactions, and will tend to be followed by large returns in either direction.

Using Ito calculus [13, 14], the equation for  $S$  follows from Eq(2) as

$$dS = \tilde{\mu}Sdt + \sigma Sd\Omega \quad (7)$$

where  $\tilde{\mu} = \mu + \frac{\sigma^2}{2}P_q^{1-q}$ . The term  $\frac{\sigma^2}{2}P_q^{1-q}$  is the noise induced drift. Remember that  $P_q$  is a function of  $\Omega$  with

$$\Omega(t) = \frac{\ln S(t)/\ln S(0) - \mu t}{\sigma} \quad (8)$$

(with  $t_0 = 0$  for simplicity.) As in the standard case (cf [2]), the noise term driving  $S$  is the same as that driving the price  $f(S)$  of a derivative of the underlying stock. It should be possible to invest one's wealth in a portfolio of shares and derivatives such that the noise terms cancel each other, yielding a risk-free portfolio, the return on which must be the risk-free rate  $r$ . This results in a generalized B-S PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 P_q^{1-q} = rf \quad (9)$$

where  $P_q(\Omega(t))$  evolves according to Eq(4). For  $q \rightarrow 1$  the standard case is recovered. This PDE depends explicitly only on the risk-free rate and the variance, not on  $\mu$ , but it does depend implicitly on  $\mu$  through  $P_q(\Omega)$ , with  $\Omega$  given by Eq(8). Therefore, to be consistent with risk-free pricing theory, we should first transform our original stochastic equation for  $S$  into a martingale before we apply the above analysis. This will not affect our results other than that  $\tilde{\mu}$  will be replaced by the risk-free rate  $r$ , ultimately eliminating the dependency on  $\mu$ . We now show how this is done.

The discounted stock price  $G = e^{-rt}S$  follows  $dG = (\tilde{\mu} - r)Gdt + \sigma Gd\Omega$  where  $d\Omega$  follows Eq(3). For there to be no arbitrage opportunities, risk-free asset pricing theory requires that this process be a martingale, which it is not due to the drift term  $(\tilde{\mu} - r)Gdt$ . One can however define an alternative driving noise  $z$  associated with an equivalent probability measure  $Q$  so that, with respect to the new noise measure, the discounted stock price has zero drift and is thereby a martingale. Explicitly,

$$dG = (\tilde{\mu} - r)Gdt + \sigma GP^{\frac{1-q}{2}}d\omega \quad (10)$$

Here,  $P$  is a non-vanishing bounded function of  $\Omega$ . With respect to the initial noise  $\omega$ ,  $\Omega$  relates to  $S$  via Eq(8). That is why for all means and purposes,

$P$  in Eq(10) is simply a function of  $S$  (or  $G$ ), and the stochastic process can be seen as a standard state-dependent Brownian one. As a consequence, both the Girsanov theorem (which specifies the conditions under which we can transform from the measure  $F$  to  $Q$ ) and the Radon-Nikodym theorem (which relates the measure  $F$  to  $Q$ ) are valid, and we can formulate equivalent martingale measures much as in the standard case [15, 16, 17]. We rewrite Eq(10) as

$$dG = \sigma G P^{\frac{1-q}{2}} dz \quad (11)$$

where the new driving noise term  $z$  is related to  $\omega$  through

$$dz = \frac{(\tilde{\mu} - r)}{\sigma P_q^{\frac{1-q}{2}}} dt + d\omega \quad (12)$$

With respect to  $z$ , we thus obtain  $dG = \sigma G d\Omega$  with  $d\Omega = P_q^{\frac{1-q}{2}} dz$  which is non other than a zero-mean Tsallis distributed generalized Wiener process, completely analogous to the one defined in Eq(3). Transforming back to  $S$  yields  $dS = rdt + \sigma S d\Omega$ . Compared with Eq(7), the rate of return  $\tilde{\mu}$  has been replaced with the risk-free rate  $r$ . This recovers the same result as in the standard asset pricing theory. Consequently, in the risk-free representation, Eq(8) becomes

$$\Omega(t) = \frac{1}{\sigma} \left( \frac{\ln S(t)}{\ln S(0)} - rt + \frac{\sigma^2}{2} \int_0^t P_q^{1-q}(\Omega(s)) ds \right) \quad (13)$$

This eliminates the dependency on  $\mu$  which we alluded to in the discussion of Eq(9). As discussed later on, by standardizing the distributions  $P_q(\Omega(s))$  we can explicitly solve for  $\Omega(t)$  as a function of  $S(t)$  and  $r$ .

Suppose that we have a European option  $C$  which depends on  $S(t)$ , whose price  $f$  is given by its expectation value in a risk-free (martingale) world as  $f(C) = E^Q[e^{-rT}C]$ . We assume the payoff on this option depends on the stock price at time  $T$  so that  $C = h(S(T))$ . After stochastic integration of Eq(11) to obtain  $S(T)$  we get

$$f = e^{-rT} E^Q \left[ h \left( S(0) \exp \left( \int_0^T \sigma P_q^{\frac{1-q}{2}} dz_s + \int_0^T \left( r - \frac{\sigma^2}{2} P_q^{1-q} \right) ds \right) \right) \right] \quad (14)$$

The key point is that the random variable  $\int_0^T P_q^{\frac{1-q}{2}} dz_s = \int_0^T d\Omega(s) = \Omega(T)$  follows the Tsallis distribution Eq(5). This gives

$$f = \frac{e^{-rT}}{Z(T)} \int_R h \left[ S(0) \exp(\sigma\Omega(T) + rT - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} + (1-q) \alpha T^{\frac{2}{3-q}} \frac{\beta(T)}{2} \sigma^2 \Omega^2(T)) \right] (1 - \beta(T)(1-q)\Omega(T)^2)^{\frac{1}{1-q}} d\Omega_T \quad (15)$$

with  $\alpha = \frac{1}{2}(3-q)((2-q)(3-q))c^{\frac{q-1}{3-q}}$ . We have utilized the fact that each of the distributions  $P(\Omega(s))$  occurring in the latter term of Eq(14) can be mapped onto the distribution of  $\Omega(T)$  at time  $T$  via the appropriate variable transformations  $\Omega(s) = \sqrt{\beta(T)/\beta(s)}\Omega(T)$ . A major difference to the standard case is the  $\Omega^2(T)$ -term which is a result of the noise induced drift. With  $q = 1$ , the standard option price is recovered [15].

Eq(15) is valid for an arbitrary payoff  $h$ . We shall evaluate it explicitly for a European call option, which gives the holder the right to buy the stock  $S$  at the strike price  $K$ , on the day of expiration  $T$ . The payoff is  $C = \max[S(T) - K, 0]$ . Only if  $S(T) > K$  will the option have value at expiration  $T$  (it will be in-the-money). The price  $c$  of such an option becomes

$$c = E^Q[e^{-rT}C] = E^Q[e^{-rT}S(T)]_D - E^Q[e^{-rT}K]_D = J_1 - J_2 \quad (16)$$

where the subscript  $D$  stands for the set  $\{S(T) > K\}$ . This condition is met if  $-\frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} + (1-q) \alpha T^{\frac{2}{3-q}} \frac{\beta(T)}{2} \sigma^2 \Omega^2 + \sigma\Omega + rT > \ln K/S(0)$ , which is satisfied for  $\Omega$  between the two roots  $s_1$  and  $s_2$  of the corresponding quadratic equation. This is a very different situation from the standard case, where the inequality is linear and the condition  $S(T) > K$  is satisfied for all values of the random variable greater than a threshold. In our case, due to the noise induced drift, values of  $S(T)$  in the risk-neutral world are not monotonically increasing as a function of the noise. As  $q \rightarrow 1$ , the larger root  $s_2$  goes toward  $\infty$ , recovering the standard case. But as  $q$  increases, the tails of the noise distribution get larger, as does the noise induced drift which tends to pull the system back. As a result we obtain

$$J_1 = S(0) \frac{1}{Z(T)} \int_{s_1}^{s_2} \exp(\sigma\Omega - \frac{\sigma^2}{2} \alpha T^{\frac{2}{3-q}} - (1-q) \alpha T^{\frac{2}{3-q}} \frac{\beta(T)}{2} \sigma^2 \Omega^2) (1 - (1-q)\beta(T)\Omega^2)^{\frac{1}{1-q}} d\Omega \quad (17)$$

$$J_2 = e^{-rT} K \frac{1}{Z(T)} \int_{s_1}^{s_2} (1 - (1-q)\beta(T)\Omega^2)^{\frac{1}{1-q}} d\Omega \quad (18)$$

The equation Eq(16) with Eq(17) and Eq(18) constitutes a closed form expression for the price of a European call. We calculated option prices for different values of the index  $q$ , and studied their properties as a function of the relevant variables such as the current stock price  $S(0)$ , the strike price  $K$ , time to expiration  $T$ , the risk free rate  $r$  and  $\sigma$ . The results obtained by our closed form pricing formula were confirmed both by implicitly solving the generalized B-S PDE Eq(9) and via Monte Carlo simulations of Eq(11). Note that American option prices can be solved numerically via Eq(9).

We compare results of the standard model ( $q = 1$ ) with those obtained for  $q = 1.5$ , which fits well to real stock returns. Figure 2 shows the difference in call price. In Figure 3, the B-S implied volatilities (which make the  $q = 1$  model match the  $q = 1.5$  results) are plotted as a function of  $K$ . The assymetric smile shape, which is more pronounced for shorter times, reproduces well-known systematic features of the “volatility smile” that appears when using the standard  $q = 1$  model to price real options. In Figure 4, the volatility smiles for actual traded options on BP and S&P 500 futures is shown together with those resulting from our model using  $q = 1.5$ . These results are encouraging, and we are currently studying a larger sample of options data. Empirical work is required to see if arbitrage opportunities can be uncovered that do not appear when the standard model is used. Another potential application will be with respect to option replication and hedging.

Acknowledgements: Fruitful discussions with Roberto Osorio and Jeremy Evnine are gratefully acknowledged.

## References

- [1] F. Black and M. Scholes, *Journal of Political Economy* **81**, 637-659, 1973
- [2] J.C. Hull, *Options, Futures, and other Derivatives*, Third Edition, Prentice-Hall, 1997
- [3] E. Eberlein, U. Keller and K. Prause, *Journal of Business* **71**, No. 3, 371-405, 1998
- [4] J.-P. Bouchaud and M. Potters, *Theory of Financial Risks*, Cambridge, 2000
- [5] L. Borland, *Phys. Rev. E* **57**, 6634, 1998

- [6] C. Tsallis, J. Stat. Phys. **52**, 479, 1988; Curado E.M.F. and Tsallis C., J. Phys. A **24** L69, 1991; **24**, 3187, 1991; **25** 1019, 1992
- [7] R. Osorio, L. Borland and C. Tsallis, in *Nonextensive Entropy: Interdisciplinary Applications* C. Tsallis and M. Gell-Mann Eds., Santa Fe Institute Studies in the Science of Complexity, Oxford, to appear
- [8] F. Michael and M.D. Johnson, cond-mat/0108017, May 2001
- [9] P. Gopikrishnan, V. Plerou, L.A. NunesAmaral, M. Meyer and H.E. Stanley, Phys. Rev. E **60**, 5305, 1999
- [10] C. Beck, Phys. Rev. Lett. **87**, 180601, 2001
- [11] C. Tsallis and D.J. Bukman, Phys. Rev. E **54**, R2197, 1996
- [12] C. Tsallis, S.V.F. Levy, A.M.C. Souza and R. Maynard, Phys. Rev. Lett. **75**, 3589, 1995
- [13] C.W. Gardiner, *Handbook of Stochastic Methods*, Second Edition, Springer, 1994
- [14] H. Risken, *The Fokker-Planck Equation*, Second Edition, Springer , 1996
- [15] B. Oksendal, *Stochastic Differential Equations*, Fifth Edition, Springer, 1998
- [16] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling* Springer , 1997 (Applications of Mathematics **36**)
- [17] S. Shreve, *Lectures on Finance* prepared by P. Chalasani and S. Jha, [www.cs.cmu.edu/~chal/shreve.html](http://www.cs.cmu.edu/~chal/shreve.html)
- [18] [www.pmpublishing.com](http://www.pmpublishing.com)



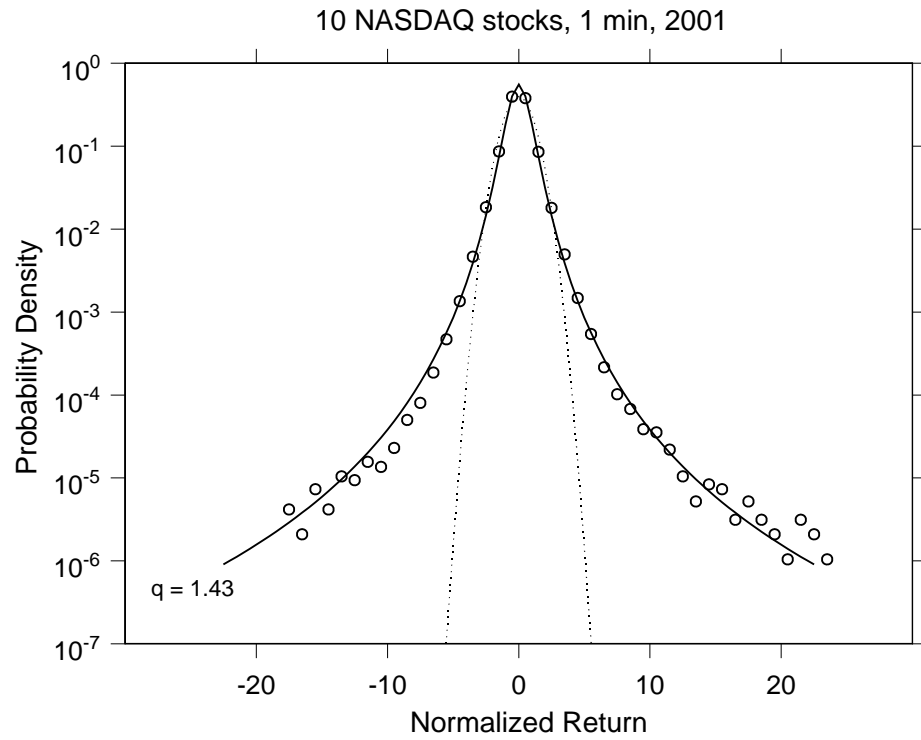


Figure 1: Distributions of log returns over 1 minute intervals for 10 high-volume stocks, normalized by the sample standard deviation. Also shown is the Tsallis distribution of index  $q = 1.43$  (solid line) which provides a good fit to the data (Figure kindly provided by R. Osorio).

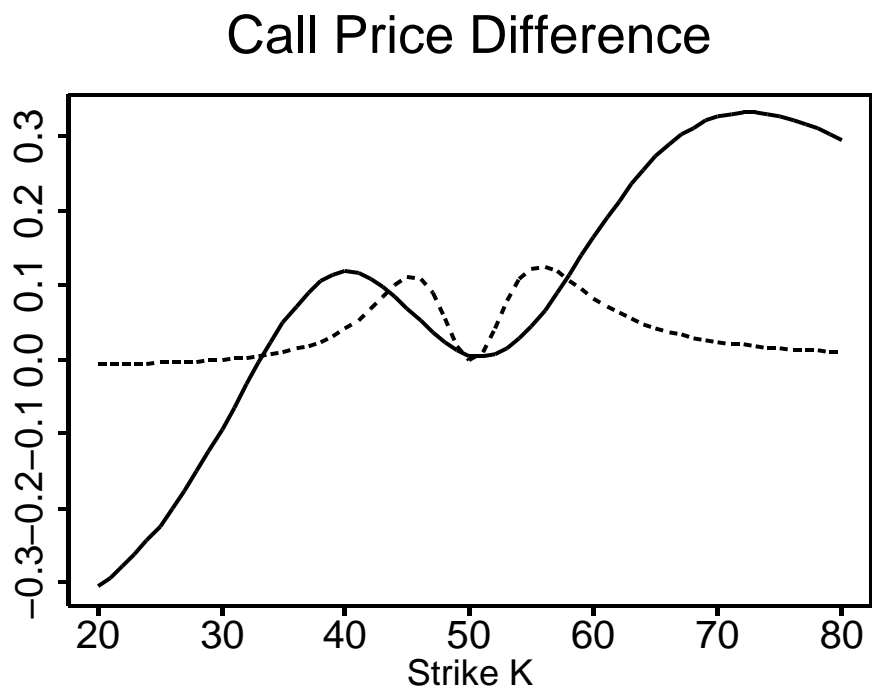


Figure 2: Calibrated so that the options are priced equally for  $S(0) = K$ , the difference between the  $q = 1.5$  model and the standard B-S model is shown, for  $S(0) = \$50$  and  $r = 0.06$ . Solid line:  $T = 0.6$  with  $\sigma = .3$  for  $q = 1$  and  $\sigma = .297$  for  $q = 1.5$ . Dashed line:  $T = 0.05$  with  $\sigma = .3$  for  $q = 1$  and  $\sigma = .41$  for  $q = 1.5$ . Times are expressed in years,  $r$  and  $\sigma$  are in annual units.

## Implied Volatility

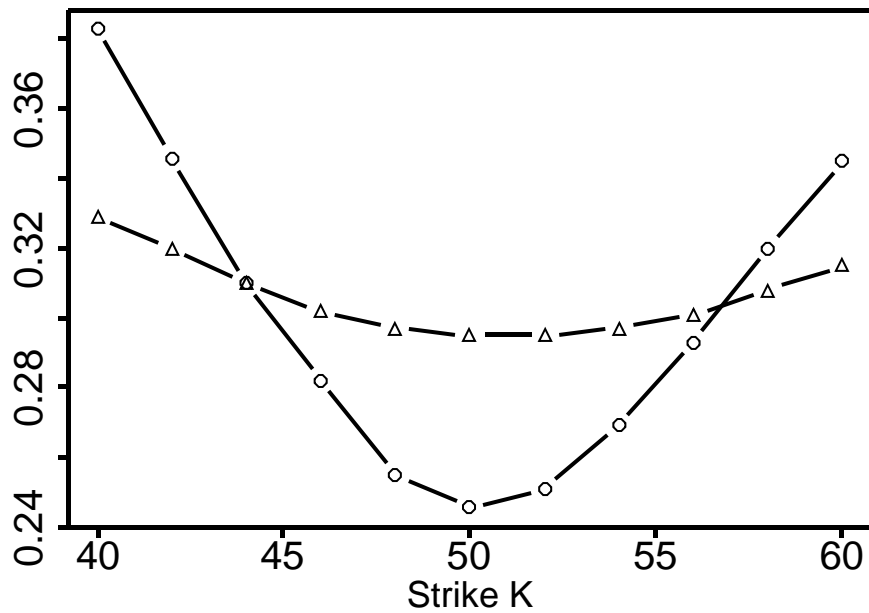


Figure 3: Using the  $q = 1.5$  model (with  $\sigma = .3$ ,  $S(0) = 50\$$  and  $r = .06$ ) to generate call option prices, one can back out the volatilities implied by a standard  $q = 1$  B-S model.  $T = 0.1$  (circles),  $T = 0.4$  (triangles).

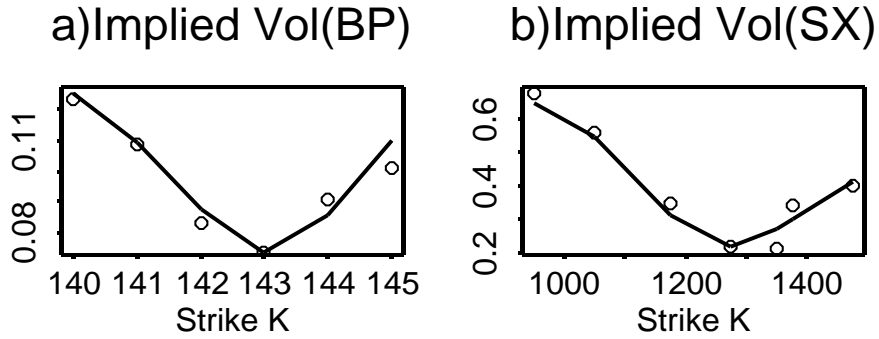


Figure 4: A comparison of option prices from a  $q = 1.5$  model and traded prices is given by a comparison of volatility smiles. a) Implied vols. for options on British Pound futures (last trading date Dec 7, 2001) [18] vs. strike (for  $S(0) = 143$ ,  $r = .065$ ,  $T = .0055$  (2 days)) (symbols); implied vols needed for a  $q = 1$  B-S model to match prices from a  $q = 1.5$  model using  $\sigma = .1445$  across all strikes (line). b) Implied vols for S&P500 futures (SX June, last trading date June 15, 2001,  $S(0) = 1275$ ,  $r = .065$ ,  $T = .027$  (10 days)) (symbols); implied vols. from  $q = 1.5$  model with  $\sigma = .3295$  (line).